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Semiclassical approximation for a nonlinear oscillator with dissipation

A. Iomin
Department of Physics, Technion, Haifa 32000, Israel
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An *S*-matrix approach is developed for the chaotic dynamics of a nonlinear oscillator with dissipation. The quantum-classical crossover is studied in the framework of the semiclassical expansion for the *S* matrix. An analytical expression for the breaking time, which is the Ehrenfest time for the dissipative system, is obtained. A correlation function of the *S*-matrix elements is studied as well.

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I. INTRODUCTION

We consider here the semiclassical dynamics of a nonlinear oscillator with dissipation. The main objective is to find the breaking time of the quantum-classical crossover for the dissipative system. In the absence of dissipation, the breaking time—namely, the Ehrenfest time—has been found [1] to scale logarithmically with respect to the Planck constant $\hbar$: $\tau_\hbar=(1/\Lambda)\ln(I_0/\hbar)$, where I_0 is a characteristic action and Λ is a Lyapunov exponent. It characterizes the exact classical-to-quantum correspondence between the Hamiltonian equation of motion and the Ehrenfest ones [1–4]. The renewed interest in this time scale is related to the extensive studies of the chaotic scattering in cavities [6] of the Loschmidt echo [7] and of the observation of an essential deviation from the logarithmic scaling for systems with phase space structures [8,9]. The nonlinear oscillator is explored to study the quantum-classical correspondence [2–5,10,11] since the Ehrenfest time scale was originally introduced in [1]. This time describes a fast (exponential) growth of quantum corrections to the classical dynamics due to chaos. In the presence of dissipation the breaking time differs from $\tau_\hbar$ since the classical dissipation changes the local instability of trajectories. The breaking time $\tau_\hbar^{(d)}$ for a dissipative web map has been obtained [9] by *c*-number projection of the Heisenberg equations on the coherent states basis. The same result has been obtained in [12] by a different method in the framework of a density matrix description. At time $\tau_\hbar^{(d)}$, the quantum corrections are of the order of 1 and destroy the (semi) classical behavior of the system. The subject of quantum dissipative chaos grew out of the the pioneering work on dissipative quantum maps [13], and various aspects of the extensive studies on quantum dissipative chaos are reflected in recent reviews [14] as well.

We show here, in the framework of an *S*-matrix approach for the chaotic dynamics of the nonlinear oscillator with a dissipation rate γ, that the breaking time $\tau_\hbar^{(d)}$ depends essentially on the ratio between the dissipation rate γ and the local instability characterized by the Lyapunov exponent Λ.

II. *S* MATRIX

The Hamiltonian of the system can be written in the non-Hermitian form

$$\mathcal{H}=\hbar\omega_\gamma a^\dagger a+\hbar^2\mu(a^\dagger a)^2-\hbar\epsilon(a^\dagger+a)\,\delta_T(t). \qquad (1)$$

The creation and annihilation operators have the commutation rule $[a,a^\dagger]=1$. The complex frequency $\omega_\gamma=\Omega-i\gamma/2$ determines the effective frequency $\omega=[\Omega^2+\gamma^2/4]^{1/2}$ in the presence of a finite width of the levels $\gamma/2$, and μ is the nonlinearity. The perturbation is a train of δ functions $\delta_T(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$, which is characterized by the amplitude ϵ and the period T. The evolution of the wave function is governed by the quantum map

$$\Psi(t+T)=\mathcal{U}(T)\Psi(t), \qquad (2)$$

where the evolution operator $\mathcal{U}(T)$ over the period T describes a free dissipative motion and then a kick. Since the decay operator commutes with the free motion one, the dissipation is applied first for sake of convenience of the notation. Therefore, the evolution operator is given by a product of the unitary evolution operator $U(T)$ and the decay operator B of the form

$$\mathcal{U}(T)\equiv\mathcal{U}=UB=e^{i\epsilon(a^\dagger+a)}e^{-i[\Omega Ta^\dagger a+\tilde{h}(a^\dagger a)^2]}e^{-\gamma Ta^\dagger a/2}. \qquad (3)$$

Here the dimensionless semiclassical parameter $\tilde{h}=\hbar\mu T$ is introduced. In what follows this parameter is small: $\tilde{h}\ll 1$. To describe the chaotic dynamics of an open system it is necessary to construct an *S* matrix. To this end we close the system by means of the complementary conditions with an incident wave $\phi_-(t)$ and an outgoing wave $\phi_+(t)$. Therefore, the quantum map (2) takes the new form [15,16]

$$\begin{pmatrix}\Psi(t+T)\\ \phi_+(t)\end{pmatrix}=\mathcal{V}\begin{pmatrix}\Psi(t)\\ \phi_-(t)\end{pmatrix}=\begin{pmatrix}\mathcal{U} & \mathcal{U}W_1\\ W_2 & S_0\end{pmatrix}\begin{pmatrix}\Psi(t)\\ \phi_-(t)\end{pmatrix}, \qquad (4)$$

where $\mathcal{V}$ is a unitary matrix: $\mathcal{V}^\dagger\mathcal{V}=1$. The operators W_1, W_2 and S_0 are determined from dissipation by solving the Schrödinger equation on the period T:

$$\Psi(t+T)=\mathcal{U}\Psi(t)-\frac{i}{\hbar}\int_{t+0}^{t+T+0}\mathcal{U}(s-t)\tilde{W}_1(s-T)\phi_-(s-T)ds. \qquad (5)$$

The operator which makes the system closed is taken in the form

$$\widetilde{W}_1(t) = W_1 \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

After the Fourier transform with respect to time we obtain that the quantum map (4) takes the form

$$e^{iET}\psi(E) = \mathcal{U}\psi(E) - \frac{i}{\hbar}\mathcal{U}W_1\phi_-(E),$$

$$\phi_+(E) = W_2\psi(E) + S_0\phi_-(E). \tag{6}$$

A relation between the incident and outgoing waves is determined by the expression

$$\phi_+(E) = S(E)\phi_-(E), \tag{7}$$

where $S(E)$ is called the S matrix [15,16]. From the definition of Eqs. (7) and (6) the S matrix reads

$$S(E) = S_0 - \frac{i}{\hbar}W_2\frac{1}{e^{-iET} - \mathcal{U}}\mathcal{U}W_1. \tag{8}$$

It is known [17] (see also [15]) that the matrix $\mathcal{V}$ can be parametrized as follows

$$\mathcal{V} = \begin{pmatrix} U\sqrt{1 - \mathcal{T}\mathcal{T}^+} & -U\mathcal{T} \\ \mathcal{T}^+ & \sqrt{1 - \mathcal{T}^+\mathcal{T}} \end{pmatrix}, \tag{9}$$

where $\mathcal{T}=i\sqrt{1-B^2}$, while $W_1=-i\hbar B^{-1}\mathcal{T}$, $W_2=-(i/\hbar)W_1^\dagger B=\mathcal{T}^\dagger$, and $S_0=B$. After the parametrization of Eq. (9), the S matrix reads

$$S(E) = B - \sqrt{1 - B^2}\frac{1}{e^{-iET} - \mathcal{U}}\mathcal{U}\sqrt{(1 - B^2)/B^2}. \tag{10}$$

III. AUTOCORRELATION FUNCTION

Now, we consider the autocorrelation function

$$R(\mathcal{E}) = \overline{\mathrm{tr}[S^\dagger(E + \mathcal{E}/2T)S(E - \mathcal{E}/2T)]} - \overline{\mathrm{tr}[S^\dagger(E)S(E)]}. \tag{11}$$

The overbar in Eq. (11) denotes the average over the quasienergy ET taken in the interval $[0,2\pi]$. Such an autocorrelation function is related to the averaged cross section [18]. The treatment of this form is analytically tractable, and in what follows we perform the semiclassical analysis for the correlation function. After simple calculations, we obtain

$$R(\mathcal{E}) = \sum_t e^{-i\mathcal{E}t}\,\mathrm{tr}[(\mathcal{U}^\dagger)^t\mathcal{U}^t - 2(\mathcal{U}^\dagger)^{t+1}\mathcal{U}^{t+1} + (\mathcal{U}^\dagger)^{t+2}\mathcal{U}^{t+2}]. \tag{12}$$

Powers t of the evolution operator $\mathcal{U}(T)$ can be formally considered as the evolution operator for an arbitrary time t—namely, $\mathcal{U}^t(T)\equiv\mathcal{U}(t)$. For the trace we take an integration over the coherent-state basis considered in the initial moment $t=0$—namely, $\mathrm{tr}(\cdots)=\int(d^2\alpha/2\pi)\langle\alpha|\cdots|\alpha\rangle$. The action of the evolution operator

$$\mathcal{U}(t) = \widehat{\exp}\left\{-i\int_0^t d\tau[\omega_\gamma a^\dagger a + \hbar\mu(a^\dagger a)^2 - \epsilon\delta_T(\tau)(a^\dagger + a)]\right\} \tag{13}$$

on the basis $|\alpha\rangle$ can be calculated analytically in the framework of the semiclassical approximation [4,5]. Here $\widehat{\exp}$ means T ordering. Applying the Stratonovich-Hubbard transformation [19] under the T ordering, one obtains, for the nonlinear term in Eq. (13),

$$\begin{aligned}\widehat{\exp}&\left[-i\hbar\mu T\int_0^t d\tau(a^\dagger a)^2/T\right] \\ &= \int\prod_\tau\frac{d\lambda(\tau)}{\sqrt{4\pi i\tilde{h}}}\exp\left(i\int_0^t d\tau\lambda^2(\tau)/4\tilde{h}\right) \\ &\quad\times\widehat{\exp}\left[-i\int_0^t d\tau\lambda(\tau)a^\dagger a\right],\end{aligned} \tag{14}$$

where $\tilde{h}=\hbar\mu T$ and $t/T\to t$ is a number of kicks represented in continuous form. We take into account that the harmonic oscillator, acting on the coherent state, changes its phase only, and the perturbation acts as a shift operator. Therefore, the wave function in the moment t has the form of the functional integral

$$\begin{aligned}\Psi(t) = U(t)|\alpha\rangle &= \int\prod_\tau(d\lambda(\tau)/\sqrt{4\pi i\tilde{h}})\exp\left[i\int_0^t d\tau\lambda^2(\tau)/4\tilde{h}\right] \\ &\times\exp\left[i\epsilon\int_0^t d\tau\delta_1(\tau)[\alpha_\lambda^*(\tau) + \alpha_\lambda(\tau)]/2\right] \\ &\times\exp\left[-(1 - e^{-\gamma T})\int_0^t d\tau\delta_1(\tau)|\alpha_\lambda(\tau)|^2\right]|\alpha_\lambda(t)\rangle,\end{aligned} \tag{15}$$

where

$$\alpha(t) = e^{-i\theta_\lambda(t)}a(t) = e^{-i\theta_\lambda(t)}\left[\alpha + i\epsilon\int_0^t d\tau\delta_1(\tau)e^{i\theta_\lambda(\tau)}\right], \tag{16}$$

$$\theta_\lambda(t) = \int_0^t d\tau[\omega_\gamma T + \lambda(\tau)] = \int_0^t d\tau[\Omega T - i\gamma T/2 + \lambda(\tau)]. \tag{17}$$

Denoting by $\beta_\lambda=-i\int_0^t d\tau\delta_1(\tau)\alpha_\lambda(\tau)$, we obtain the following expression for the trace:

$$\begin{aligned}\mathcal{M}(t) &= \int\frac{d^2\alpha}{2\pi}\langle\alpha|\mathcal{U}^\dagger(t)\mathcal{U}(t)|\alpha\rangle = \int\frac{d^2\alpha}{2\pi}\int\prod_\tau\frac{d\lambda_1(\tau)d\lambda_2(\tau)}{4\pi\tilde{h}} \\ &\times\exp\left[\frac{i}{4\tilde{h}}\int_0^t d\tau[\lambda_1^2(\tau) - \lambda_2^2(\tau)]\right] \\ &\times\exp\left[i\,\mathrm{Im}(\alpha_{\lambda_2}^*\alpha_{\lambda_1} - \epsilon\beta_{\lambda_1} - \epsilon\beta_{\lambda_2}^*) - \frac{1}{2}|\alpha_{\lambda_1} - \alpha_{\lambda_2}|^2\right] \\ &\times\exp\left[-(1 - e^{-\gamma T})\int_0^t d\tau\delta_1(\tau)[|\alpha_{\lambda_1}(\tau)|^2+|\alpha_{\lambda_2}(\tau)|^2]/2\right].\end{aligned} \tag{18}$$

In the limit $\tilde{h} \ll 1$, the expression for the trace $\mathcal{M}(t)$ is strongly simplified and evaluated analytically. Following [5], we perform the linear transform $\lambda_1 = 2\mu + \tilde{h}\nu/2$, $\lambda_2 = 2\mu - \tilde{h}\nu/2$, where the Jacobian equals $2\tilde{h}$. After the variables change, we obtain from Eqs. (16) and (17) the following semiclassical expressions for the second exponential in Eq. (18):

$$\alpha^*_{\lambda_2}(t)\alpha_{\lambda_1}(t) - \epsilon\beta_{\lambda_1} - \epsilon\beta^*_{\lambda_2} \approx - \int_0^t d\tau [i\tilde{h}\nu(\tau) + \gamma T] e^{-\gamma T\tau} |a(\tau)|^2, \qquad (19)$$

$$|\alpha_{\lambda_1} - \alpha_{\lambda_2}|^2 \approx \left| \int_0^t d\tau \tilde{h}\nu(\tau) e^{-\gamma T\tau/2} a(\tau) \right|^2. \qquad (20)$$

Here $e^{-\gamma T\tau/2}a(\tau)$ is defined in Eqs. (16) and (17) for $\nu \equiv 0$. Now we perform integration over $\nu(\tau)$ and $\mu(\tau)$ in the classical limit, neglecting the term of the order of $\tilde{h}^2$ defined in Eq. (20). The integral over ν yields $\Pi_\tau 2\pi\delta(\mu - \tilde{h}|e^{-\gamma T\tau/2}a(\tau)|^2)$, and it leads to the exact integration over μ as well. Finally, we obtain, for Eq. (18),

$$\mathcal{M}(t) = \int \frac{d^2\alpha}{2\pi} \exp\left[- \frac{(1 - e^{-\gamma T})}{\tilde{h}} \int_0^t d\tau \delta_1(\tau) I_{cl}(\tau, \alpha, \alpha^*) - (\gamma T/\tilde{h}) \int_0^t d\tau I_{cl}(\tau, \alpha, \alpha^*) \right], \qquad (21)$$

where we denote $\alpha(\tau) = \sqrt{I(\tau)/\tilde{h}} e^{-i\theta(\tau)}$ [see also Eqs. (16) and (17)] and $I(\tau) \equiv I_{cl}(\tau, \alpha, \alpha^*)$. To evaluate the integrals in the exponential in Eq. (21) we take into account that the dynamics takes place on a chaotic attractor. Classical dynamics on the attractor is determined by the map $\hat{T}(I, \theta) \to (I, \theta)$. In the action-angle (I, θ) variables this map has a very complicated form, because the perturbation in the classical counterpart of the Hamiltonian (1) is a function of both I and θ. To obtain a crude criterion of the existance of the chaotic attractor, we explore the following approximation of the map $\hat{T}$:

$$I_{\tau+1} = e^{-\gamma T}[I_\tau + 2\epsilon\sqrt{I_\tau} \sin \theta_\tau + \epsilon^2],$$

$$\theta_{\tau+1} = \theta_\tau + \Omega T + 2\mu T I_{\tau+1}. \qquad (22)$$

Despite being an approximation of an exact analysis of [1], the map is detailed enough to obtain the local instability condition in the form

$$|\partial\theta_{\tau+1}/\partial\theta_\tau - 1| \sim 4\mu\epsilon T \sqrt{I_\tau} e^{-\gamma T} = K e^{-\gamma T} > 1.$$

Since the minimum of the action on the attractor is $\min(I_\tau) \geq \epsilon^2$, a rough estimation of the chaos control parameter K can be obtained by a substitution $I_\tau \sim \epsilon^2$. This corresponds to the limit cycle with the minimum phase-space volume of the order of ϵ^2. In this case, the criterion of the chaotic attractor is

$$K = 4\mu\epsilon^2 T > e^{\gamma T} > 1. \qquad (23)$$

It is convenient to present the action as a sum of two terms, $I_{cl}(\tau) = \tilde{h}|\alpha|^2 e^{-\gamma T\tau} + \tilde{I}_{cl}(\tau)$, where the first term relates to the initial conditions and the second is a classical action with zero initial conditions $\tilde{I}_{cl}(\tau = 0) = 0$. It should be stressed that when condition (23) is fulfilled, the chaotic attractor takes place in a fixed and finite part of phase space and the classical action is limited for any time τ: $\epsilon^2 < I_{cl}(\tau) < \max(I_\tau)$. Therefore, one can apply the mean-value theorem for integration of the classical action in Eq. (21). It gives $\int_0^t \tilde{I}_{cl}(\tau) d\tau \sim \int_0^t \delta_1(\tau)\tilde{I}_{cl}(\tau) d\tau \sim t\gamma T\langle\tilde{I}\rangle$, where $\langle\tilde{I}\rangle$ is a characteristic average action on the attractor, which is independent of the initial conditions α, α^* as well. It is convenient to rewrite it in the form $\langle\tilde{I}\rangle = \langle I\rangle/(1 + \gamma T - e^{-\gamma T})$. Integration of the first term in Eq. (21) gives $2|\alpha|^2$ for $t \gg 1/\gamma T$. Using this crude but reasonable estimation of Eq. (21), we obtain the following expression for the trace:

$$\mathcal{M}(t) \approx \int \frac{d^2\alpha}{2\pi} \exp[-2|\alpha|^2 - \gamma T t\langle I\rangle/\tilde{h}] \propto \exp[-\gamma T t\langle I\rangle/\tilde{h}]. \qquad (24)$$

Inserting this result into Eq. (12), we obtain an expression for the correlation function in the form

$$R(\mathcal{E}) = \frac{(1 - e^{-\gamma T})^2}{1 - \exp(-i\mathcal{E} - \gamma T\langle I\rangle)}. \qquad (25)$$

It is worth mentioning that the autocorrelation function $R(\mathcal{E})$ related to an averaged cross section corresponds to the Ericson fluctuations (see e.g. [18]). Considering that $\mathcal{E}$ are small [20], we take into account only the first two terms in the expansion of the exponential in the denominator in Eq. (25). Following [20,21] we calculate the correlation function $|R(\mathcal{E})|^2$ in the normalized form $|\mathcal{R}|^2 = |R(\mathcal{E})|^2/|R(0)|^2$. Finally, we arrive at the expression

$$|\mathcal{R}|^2 \approx \frac{1}{1 + (\mathcal{E}/\mathcal{G})^2}, \qquad (26)$$

where $\mathcal{G} = e^{\gamma T\langle I\rangle}$. The Lorentzian describes the distribution of the Ericson fluctuations; see, e.g., [18,20,21].

IV. BREAKING TIME

The important point of the consideration is that the term of the order of $\tilde{h}^2$—namely, $\tilde{h}^2|\int d\tau\nu(\tau)a(\tau)|^2$—is neglected. This means that the quantum chaotic attractor is well described by classical equations of motion—namely, by map (22)—and leads to the restriction on time which characterizes the breaking time of classical-to-quantum correspondence. It is the Ehrenfest time which specifies the validity condition of the performed semiclassical approximation when the S matrix is random with corresponding correlations of the matrix elements in the Ericson regime of Eq. (26). To evaluate the omitted $O(\tilde{h}^2)$ term in $\mathcal{M}(t)$, we take this into account at the integration over $\nu(\tau)$ in Eq. (18). To this end we use the auxiliary expression

$$\exp\left[-\frac{\tilde{h}^2}{2}\left|\int_0^t d\tau\nu(\tau)e^{-\gamma T\tau/2}a(\tau)\right|^2\right]$$
$$=\frac{2}{\pi\tilde{h}}\int d^2\xi e^{-2|\xi|^2/\tilde{h}}$$
$$\times\exp\left[-i\frac{\sqrt{\tilde{h}}}{2}\,\mathrm{Re}\,\xi^*\int_0^t d\tau\nu(\tau)e^{-\gamma T\tau/2}a(\tau)\right]. \quad (27)$$

Substituting Eqs. (19), (20), and (27), into Eq. (18), one obtains the following contribution to the trace:

$$\frac{2}{\tilde{h}\pi}\int d^2\xi e^{-2|\xi|^2/\tilde{h}}\int\prod_\tau\frac{d\mu(\tau)d\nu(\tau)}{2\pi}$$
$$\times\exp\left[-i\int_0^t d\tau\nu(\tau)\mu(\tau)\right]$$
$$\times\exp\left[i\tilde{h}\int_0^t d\tau\nu(\tau)[|e^{-\gamma T\tau/2}a(\tau)+\xi|^2-|\xi|^2]\right]. \quad (28)$$

The functional integration over $\nu(\tau)$ is exact and results in the δ function in μ: $\Pi_\tau 2\pi\delta(\mu-\tilde{h}|e^{-\gamma T\tau/2}a(\tau)+\xi|^2-|\xi|^2)$. Hence the integration over $\mu(\tau)$ is also exact. Performing these integrations we obtain from Eqs. (18) and (28) that

$$\mathcal{M}=\frac{1}{\tilde{h}\pi^2}\int d^2\alpha\int d^2\xi e^{-2|\xi|^2/\tilde{h}}$$
$$\times\exp\left[-\int_0^t\frac{d\tau}{\tilde{h}}G(\tau)\overline{I_{cl}}(\alpha,\alpha^*,\xi,\xi^*,\tau)\right], \quad (29)$$

where the expression

$$\overline{I_{cl}}(\alpha,\alpha^*,\xi,\xi^*,\tau)\equiv I_{cl}(\Omega T-2|\xi|^2,e^{-\gamma T\tau/2}a(\tau)+\xi,e^{-\gamma T\tau/2}a^*(\tau)+\xi^*)$$

is the classical action with shifted initial conditions and linear frequency. The amplitudes $a(\tau)$ and $a^*(\tau)$ are determined in Eq. (16), while $G(\tau)=[\gamma T+(1-e^{-\gamma T})\delta_1(\tau)]$. Expanding the last exponential in Eq. (29) in the Taylor series in ξ and ξ^* and taking into account that

$$(2/\pi\tilde{h})\int d^2\xi e^{-2|\xi|^2/\tilde{h}}\xi^p\xi^{*q}=\sqrt{(\tilde{h}/2)^{p+q}}\sqrt{p!q!}\,\delta_{p,q},$$

we obtain the trace of Eq. (29) in the form of the Taylor expansion in the semiclassical parameter $\tilde{h}$:

$$\mathcal{M}=\int\frac{d^2\alpha}{2\pi}\sum_{n,l}\frac{(n+l)!}{(n!)^2l!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\frac{\partial^l}{\partial(\Omega T)^l}(-2)^l(\tilde{h}/2)^{n+l}$$
$$\times\exp\left[-(1/\tilde{h})\int_0^t d\tau G(\tau)\overline{I_{cl}}(\tau,\alpha,\alpha^*)\right]. \quad (30)$$

The validity of Eq. (21), which is the zero order of $\tilde{h}$, allows one to neglect all the rest, which are higher order in $\tilde{h}$ in the Taylor series of Eq. (30). It is clear that if the criterion (23) is fulfilled resulting in the existence of the chaotic attractor, the strongest contribution to the expansion for the same order of $\tilde{h}$ is due to the second derivatives $(\partial^2/\partial\alpha\partial\alpha^*)$. Therefore, one obtains the chain of derivatives

$$\frac{\partial\overline{I(\tau)}}{\partial\alpha}\sim\sqrt{\tilde{h}/I_0}\left(\frac{\partial\overline{I(\tau)}}{\partial\theta_0}\right)\sim\sqrt{\tilde{h}/I_0}\left(\frac{\partial\overline{I(\tau)}}{\partial\theta_{\tau-1}}\right)\left(\frac{\partial\theta_{\tau-1}}{\partial\theta_{\tau-2}}\right)\cdots\left(\frac{\partial\theta_1}{\partial\theta_0}\right). \quad (31)$$

From the classical map (22) and Eq. (23) we have that $(\partial\theta_j/\partial\theta_{j-1})\sim Ke^{-\gamma T}>1$. Therefore, the chain (31) yields

$$\frac{\partial\overline{I(\tau)}}{\partial\alpha}\sim\sqrt{\tilde{h}/I_0}K^\tau e^{-\tau\gamma T}=\sqrt{\tilde{h}/I_0}\exp[\tau(\ln K-\gamma T)]. \quad (32)$$

The same expression is for $\partial\overline{I(\tau)}/\partial\alpha^*$. Finally, the strongest contribution to the term of the first order of $\tilde{h}$ is

$$D(I,I)=\frac{I_0}{\tilde{h}}\frac{\partial\overline{I(\tau)}}{\partial\alpha}\frac{\partial\overline{I(\tau)}}{\partial\alpha^*}=\exp[2\tau(\Lambda-\gamma T)], \quad (33)$$

where $\Lambda=\ln K$ is the Lyapunov exponent. The validity condition of the performed approximation is $D(I,I)<I_0\langle I\rangle/\tilde{h}^2\sim(I_{cl}/\tilde{h})^2$. It yields the validity condition for the time scale, which is a breaking time between classical and quantum dynamics for the nonlinear kicked oscillator in the presence of dissipation. It reads

$$\tau_\hbar^{(d)}=\ln(I_{cl}/\tilde{h})/(\Lambda-\gamma T). \quad (34)$$

It follows from Eq. (23) that the denominator in Eq. (34) is always positive, but it can be arbitrarily small. The situation, when $\ln K-\gamma T$ is very small, was called in [22] "the dying attractor." In this case $\tau_\hbar^{(d)}$ is arbitrarily large but finite (see also [9]).

V. CONCLUSION

The semiclassical approximation for the S matrix is developed for the quantum chaotic attractor of the nonlinear oscillator. An analytical expression for the breaking time $\tau_\hbar^{(d)}$ of classical-to-quantum correspondence is obtained. For $\gamma=0$ it coincides with the Ehrenfest time $\tau_\hbar$. The result of Eq. (34) expresses the fundamental correspondence principle. It establishes relations between the main parameters—namely, the dimensionless semiclassical parameter $\tilde{h}$, the global chaos parameter K, and the dissipation rate γ—which determine the quantum dynamics of the system with the non-Hermitian Hamiltonian.

An important point of the semiclassical consideration is that the Ehrenfest time is the result of the Taylor expansion; namely, it results from the condition of the Taylor series being convergent. This approach is absolutely different from the standard semiclassical expansion which is an asymptotic one. This has been the subject of wide discussion since the seminal paper of [1]. (Also see a recent discussion [23,24] and references therein.) The essential difference between these two semiclassical approaches is as follows. The semiclassical consideration on the Ehrenfest time scale $\tau_\hbar$ or $\tau_\hbar^{(d)}$

is a situation when quantum dynamics is well described by the classical equations of motion. In this case, the Ehrenfest time scales logarithmically with respect to $\tilde{h}$. Conversely, semiclassical Wentzel-Kramers-Brillouin (WKB) approximation takes into account a quantum interference effect that leads to the breaking time scales as the (inverse) power law in $\tilde{h}$. It could be obtained from Eq. (30) by resummation of the Taylor expansion that corresponds to the semiclassical consideration beyond the Ehrenfest time [5,11]. It is an essential advantage of the *S*-matrix consideration that enables one to consider a quantum chaotic attractor beyond the Ehrenfest time. In this case, this *S*-matrix approach could be an effective way to study chaotic attractors where a possible phase-space structure, as a cantor set, could be studied in the framework of the semiclassical consideration on the finite time scale of Eq. (34) or beyond it.

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